A new attempt towards the unification of space-time and internal gauge symmetries[1]

V. Aldaya[2,3] and E. Sánchez-Sastre[2,3]

Abstract

The neat formulation that describes the gauge interactions associated with internal symmetries is extended to the case of a simple, yet non-trivial, symmetry group structure which mixes gravity and electromagnetism by associating a gauge symmetry with a central extension of the Poincaré group.

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[1]Work partially supported by the DGICYT.

[2]Instituto de Astrofisica de Andalucia, Apartado Postal 3004, Granada 18080, Spain

[3]Instituto de Física Teórica y Computacional Carlos I, Facultad de Ciencias, Universidad de Granada, Campus de Fuentenueva, Granada 18002, Spain

1 Introduction

The notion of *gauge* symmetry is traced back to Weyl[1] with respect to the invariance of a system under scale ("gauge") transformations depending on the particular space-time point. However, nowadays in modern physics the term "gauge" has nothing to do with scale transformations but with the whole picture that describes the fundamental interactions. In the standard Lagrangian formalism, promoting a given underlying rigid symmetry to a "local" one requires the introduction of a connection which is interpreted as a potential providing the corresponding gauge interaction. This is essentially the formulation of the so-called Minimal Coupling Principle. Internal gauge invariance has successfully led to the electromagnetic interaction associated with $U(1)$, electroweak interactions associated with $(SU(2)\otimes U(1))/Z_2$, and finally to the strong interaction associated with colour $SU(3)$. As an extra bonus of gauge theory, the association of interactions with groups translates the problem of unification of forces to that of finding rigid symmetry groups containing older ones as non-trivial (not as a direct product) subgroups. Although the final choice of a "grand unification group" for internal symmetry, of the type $SU(5)$[2] or $SO(10)$[3], still remains to be found, the actual problems for achieving such a result are of a phenomenological nature[4].

The case of the gravitational interaction understood as some sort of gauge theory is a question which was firstly considered by Utiyama (1956)[5] and later by Kibble (1961)[6]. After these pioneer papers, much effort has been devoted to achieving a clear understanding of the gauge nature of the gravitational field (see among others [7−31]), although fully disconnected from other interactions. The unification of gravity and the other interactions would have supposedly required the non-trivial mixing of the space-time group and some internal symmetry, a task explicitly "forbidden" long ago by the so-called "no-go" theorems by O'Raifeartaigh, Coleman, Mandula, Michel, etc. [32 − 37]. The "no-go" theorems state that there is no finite-dimensional Lie group containing the Poincaré group, acting as diffeomorphisms of the Minkowski space-time, and any internal $SU(n)$ group, except for the direct product. In this paper, we shall bypass no-go theorems in a subtle way by replacing the Poincaré group with the space-time symmetry of the relativistic quantum particle, i.e. a central extension of the Poincaré group by $U(1)$ (see [38] and references there in). The proposed symmetry has been succesfully used in a Group Approach to Quantization (GAQ) [39] to describe the (classical) particle-mechanics analog of the present problem [38]. GAQ was originally formulated as a group-theoretical quantization scheme designed for obtaining the quantum dynamics of a physical system out of a given centrally extended Lie group. However, it also describes naturally the classical limit in the Hamilton-Jacobi picture.

The paper is organized as follows. Sec. II is devoted to the general structure of gauge theory including space-time symmetries. In Sec. III we present the gauging of the centrally extended Poincaré group giving rise to the new phenomenon of an extra coupling constant mixing non-trivially the geodesic and the Lorentz forces.

2 Brief review of the general structure of gauge theory for internal and space-time symmetries

2.1 Internal symmetries

Let us consider a matter Lagrangian density $\mathcal{L}_{matt}(\varphi^\alpha, \varphi^\alpha_{,\mu})$[1] depending on the matter fields φ^α and their first-order derivatives $\varphi^\alpha_{,\mu} \equiv \frac{\partial \varphi^\alpha}{\partial x^\mu}$. Let us assume that the matter action

$$\mathcal{S} = \int \mathcal{L}_{matt}(\varphi^\alpha, \varphi^\alpha_{,\mu})\ d^4x \tag{1}$$

is invariant under a global (rigid) Lie group of internal symmetry. The infinitesimal transformation of the matter fields (associated with each group generator with index (a)) under G is supposed to be

$$\delta_{(a)}\varphi^\alpha = X^\alpha_{(a)\beta}\varphi^\alpha, \tag{2}$$

where $X^\alpha_{(a)\beta}$ denotes a matrix realization of the infinitesimal action of the Lie group generators, satisfying the commutation relations

$$(X_{(b)}X_{(a)} - X_{(a)}X_{(b)})^\alpha_\beta = C^c_{ab}X^\alpha_{(c)\beta}, \tag{3}$$

the C^c_{ab} being the structure constants of the group. Hence, the global invariance condition of the action reads:

$$\delta^{global}_{(a)}\mathcal{L}(\varphi^\alpha, \varphi^\alpha_{,\mu}) = X^\alpha_{(a)\beta}\varphi^\beta \frac{\partial \mathcal{L}}{\partial \varphi^\alpha} + X^\alpha_{(a)\beta}\varphi^\beta_{,\mu}\frac{\partial \mathcal{L}}{\partial \varphi^\alpha_{,\mu}} = 0. \tag{4}$$

Let us consider the ("current", "local" or) gauge group $G(M)$, i.e. a group G with parameters depending on the space-time points. The corresponding Lie algebra is the tensor product $\mathcal{F}(M) \otimes \mathcal{G}$ where $\mathcal{F}(M)$ is the multiplicative algebra of real analytic functions (which will be denoted in the sequel by $f^{(a)}$) on M, and $\mathcal{G}$ is the Lie algebra of the Lie group G. Obviously, the action is not invariant under $G(M)$:

$$\delta\mathcal{L}(\varphi^\alpha, \varphi^\alpha_{,\mu}) = f^{(a)}\delta^{global}_{(a)}\mathcal{L}(\varphi^\alpha, \varphi^\alpha_{,\mu}) + X^\alpha_{(a)\beta}\varphi^\beta \frac{\partial f^{(a)}(x)}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \varphi^\alpha_{,\mu}} = X^\alpha_{(a)\beta}\varphi^\beta \frac{\partial f^{(a)}(x)}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \varphi^\alpha_{,\mu}} \neq 0. \tag{5}$$

[1]The index notation throughout this paper is the following: we shall use the first half of the Greek alphabet $\alpha, \beta, \gamma, ...(= 1, ..., N)$ to denote the internal components (the representation indices) of the matter fields, the second half of the Greek alphabet $\mu, \nu, \lambda, ...(= 0, ..., 3)$ will denote space-time indices (the space indices running from 1 to 3 will be denoted with letters from the middle of the Latin alphabet $i, j, k, ...$). Finally we shall use the first half of the Latin alphabet in brackets $(a), (b), (c), ...(= 1, ..., dimG)$ to denote the group indices. We emphasize that the brackets in the group indices by no means are related to symmetrization or antisymmetrization.

Note that $\delta^{global}_{(a)}\mathcal{L}(\varphi^\alpha, \varphi^\alpha_{,\mu}) = 0$ by hypothesis. In order to restore the invariance under $G(M)$ we have to introduce new compensating fields (usually known as Yang-Mills fields) $A^{(a)}_\mu$ with the usual transformation law of a connection under $G(M)$:

$$\delta A^{(a)}_\mu = f^{(b)} C^a_{bc} A^{(c)}_\mu + \frac{\partial f^{(a)}}{\partial x^\mu}\,, \tag{6}$$

The new fields $A^{(a)}_\mu$ modify the behaviour of the original Lagrangian of matter so that we have to find, on the one hand, the expression for the new Lagrangian $\widehat{\mathcal{L}}_{matt}$ containing the matter fields and their interaction with the new compensating fields $A^{(a)}_\mu$ and, on the other, the free Lagrangian $\mathcal{L}_0$ corresponding to the new fields, which should depend on the new field variables and their first derivatives, i.e. $A^{(a)}_\mu$, $A^{(a)}_{\nu,\sigma} \equiv \frac{\partial A^{(a)}_\nu}{\partial x^\sigma}$. It is well-known that the solution to this question is given by the Minimal Coupling Prescription, which states that *The new Lagrangian describing the matter fields as well as their interaction with the new compensating fields* $A^{(a)}_\nu$ *has the form*

$$\widehat{\mathcal{L}}_{matt}(\varphi^\alpha, \varphi^\alpha_{,\mu}, A^{(a)}_\nu) \equiv \mathcal{L}_{matt}(\varphi^\alpha, \varphi^\alpha_{,\mu} - A^{(a)}_\mu X^\alpha_{(a)\beta}\varphi^\beta). \tag{7}$$

In other words, the matter Lagrangian incorporating the interaction terms is obtained from the original one by replacing all derivatives of the matter fields with covariant derivatives.

The introduction of the gauge (compensating) fields naturally leads to considering a new action accounting also for the dynamics of these new fields with Lagrange density $\mathcal{L}_0(A^{(a)}_\mu, A^{(b)}_{\nu,\sigma})$:

$$\mathcal{S}' = \int (\widehat{\mathcal{L}}_{matt} + \mathcal{L}_0) d^4x. \tag{8}$$

Since $\int \widehat{\mathcal{L}}_{matt}\, d^4x$ is invariant under $G(M)$, imposing the invariance of $\mathcal{S}'$ requires the invariance of $\int \mathcal{L}_0\, d^4x$ itself. That is, the free Lagrangian $\mathcal{L}_0$, containing the new compensating fields and their first derivatives, must be invariant under the current group $G(M)$:

$$\begin{aligned}\delta\mathcal{L}_0(A^{(c)}_\mu, A^{(b)}_{\nu,\sigma}) &= \left(f^{(b)} C^a_{bc} A^{(c)}_\mu + \frac{\partial f^{(a)}}{\partial x^\mu}\right)\frac{\partial \mathcal{L}_0}{\partial A^{(a)}_\mu} \\ &+ \left(f^{(b)} C^a_{bc} A^{(c)}_{\mu,\nu} + C^a_{bc} A^{(c)}_\mu \frac{\partial f^{(b)}}{\partial x^\nu} + \frac{\partial^2 f^{(a)}}{\partial x^\nu \partial x^\mu}\right)\frac{\partial \mathcal{L}_0}{\partial A^{(a)}_{\mu,\nu}} = 0\,. \end{aligned} \tag{9}$$

This requirement of gauge invariance of $\mathcal{L}_0$ implies that *the necessary condition for* $\mathcal{L}_0$ *to be invariant under the current group* $G(M)$ *is that* $\mathcal{L}_0$ *depends on the fields* $A^{(a)}_\mu$ *and their "derivatives"* $A^{(a)}_{\mu,\nu}$ *only through the specific combination:*

$$F^{(a)}_{\mu\nu} \equiv A^{(a)}_{\mu,\nu} - A^{(a)}_{\nu,\mu} - \frac{1}{2} C^a_{bc}(A^{(b)}_\mu A^{(c)}_\nu - A^{(b)}_\nu A^{(c)}_\mu)\,, \tag{10}$$

which is traditionally called the "curvature" of the "connection" $A^{(a)}_\mu$ (see again the end of this subsection).

It should be remarked that the actual dependence of $\mathcal{L}_0$ on the tensor F is not fixed and must be chosen with the help of extra criteria, for example the invariance under the rigid Poincaré group. In particular, to account for the standard Yang-Mills equations the Lagrangian must be of the form

$$\mathcal{L}_0 \sim \sum_{a=1}^{dimG} F_{\mu\nu}^{(a)} F_{\sigma\rho}^{(a)} \eta^{\sigma\mu} \eta^{\rho\nu} \,. \tag{11}$$

Introducing the notation (spin connection)

$$\Gamma^{\alpha}_{\mu\beta} \equiv A_{\mu}^{(a)} X^{\alpha}_{(a)\beta} \,, \tag{12}$$

and taking into account the commutation relations (3), the tensor $F_{\mu\nu}^{(a)}$ can be turned into a curvature tensor:

$$\begin{aligned} R^{\alpha}_{\mu\nu\beta} \equiv F_{\mu\nu}^{(a)} X^{\alpha}_{(a)\beta} &= \partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} - \frac{1}{2} C^{a}_{bc} (A_{\mu}^{(b)} A_{\nu}^{(c)} (X_{(a)})^{\alpha}_{\beta} - A_{\nu}^{(b)} A_{\mu}^{(c)} (X_{(a)})^{\alpha}_{\beta}) \\ &= \partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} - (\Gamma^{\alpha}_{\mu\gamma} \Gamma^{\gamma}_{\nu\beta} - \Gamma^{\alpha}_{\nu\beta} \Gamma^{\gamma}_{\mu\beta}) \,. \end{aligned} \tag{13}$$

The content of this subsection summarizes briefly the general scheme of the well-known formulation of gauge theory associated with internal symmetry groups. Subtle questions such as the Higgs-Kibble mechanism via spontaneous symmetry breaking are not considered in the present paper. However, in a forthcoming work [40] we shall propose an alternative mass-generating mechanism for the gauge vector bosons which is based essentially on the introduction of the group parameters in the theory as dynamical fields.

2.2 Space-time symmetries

In this subsection we generalize the previous one to the case in which the rigid group also acts on the space-time. The infinitesimal transformation of the space-time coordinates and the matter fields is taken to be of the form

$$\delta_{(a)} x^{\mu} = X^{\mu}_{(a)} \tag{14}$$

$$\delta_{(a)} \varphi^{\alpha} = X^{\alpha}_{(a)\beta} \varphi^{\beta}, \tag{15}$$

where $X^{\mu}_{(a)}$ is in general a function of the position. As in the internal symmetry case the starting point of the theory is the hypothesis of global invariance of the matter action, i.e.

$$X^{\mu}_{(a)} \frac{\partial \mathcal{L}_{matt}}{\partial x^{\mu}} + X^{\alpha}_{(a)\beta} \varphi^{\beta} \frac{\partial \mathcal{L}_{matt}}{\partial \varphi^{\alpha}} + (X^{\alpha}_{(a)\beta} \varphi^{\beta}_{,\mu} - \varphi^{\alpha}_{,\nu} \frac{\partial X^{\nu}_{(a)}}{\partial x^{\mu}}) \frac{\partial \mathcal{L}_{matt}}{\partial \varphi^{\alpha}_{,\mu}} + \mathcal{L}_{matt} \partial_\mu X^{\mu}_{(a)} = 0. \tag{16}$$

It is remarkable the appearance of the divergence of the action of the group on the space-time coordinates $\partial_\mu X^{\mu}_{(a)}$, a term which was absent for the internal symmetry case. This is a consequence of the variation of the integration volume: $\delta_{(a)} d^4x = \partial_\mu X^{\mu}_{(a)} d^4x$.

Let us construct an invariant action under the local (gauge) space-time group generated by:

$$
\begin{aligned}
f^{(a)}(x)\delta_{(a)}x^{\mu} &= f^{(a)}(x)X^{\mu}_{(a)} &(17)\\
f^{(a)}(x)\delta_{(a)}\varphi^{\alpha} &= f^{(a)}(x)X^{\alpha}_{(a)\beta}\varphi^{\beta}. &(18)
\end{aligned}
$$

It is worth realizing that the dependence of the space-time components of the generators on x^{μ} through $X^{\mu}_{(a)}$ is not "gauge". The gauge dependence on x^{μ} arises from the fact that these generators are multiplied by arbitrary functions $f^{(a)}(x)$.

The construction of a gauge invariant Lagrangian density requires the introduction of new fields. Apart from compensating fields $\mathcal{A}^{(a)}_{\nu}$ analogous to those of internal symmetries, there will be additional compensating fields k^{ν}_{μ} (tetrad fields) related to the group action on the space-time. The corresponding transformation laws of the compensating fields under $G(M)$ read:

$$
\begin{aligned}
\delta\mathcal{A}^{(a)}_{\mu} &= f^{(b)}C^{a}_{bc}\mathcal{A}^{(c)}_{\mu} + k^{\nu}_{\mu}\frac{\partial f^{(a)}}{\partial x^{\nu}} - f^{(b)}\mathcal{A}^{(a)}_{\sigma}\frac{\partial X^{\sigma}_{(b)}}{\partial x^{\mu}} &(19)\\
\delta k^{\nu}_{\mu} &= X^{\nu}_{(a)}k^{\sigma}_{\mu}\frac{\partial f^{(a)}}{\partial x^{\sigma}} + f^{(a)}\Big(k^{\sigma}_{\mu}\frac{\partial X^{\nu}_{(a)}}{\partial x^{\sigma}} - k^{\nu}_{\sigma}\frac{\partial X^{\sigma}_{(a)}}{\partial x^{\mu}}\Big). &(20)
\end{aligned}
$$

Inverse fields of k^{ν}_{μ} will be denoted by q^{μ}_{σ}, so that

$$
\begin{aligned}
k^{\nu}_{\mu}q^{\mu}_{\sigma} &= \delta^{\nu}_{\sigma} &(21)\\
k^{\nu}_{\mu}q^{\sigma}_{\nu} &= \delta^{\sigma}_{\mu}. &(22)
\end{aligned}
$$

Following similar steps to those in the internal case we can establish a generalized Minimal Coupling Prescription by saying that *the new Lagrangian describing the matter fields as well as their interaction with the compensating fields* $\mathcal{A}^{(a)}_{\nu}$, k^{ν}_{μ} *has the form*

$$
\widehat{\mathcal{L}}_{matt}(\varphi^{\alpha}, \varphi^{\alpha}_{,\mu}, \mathcal{A}^{(a)}_{\nu}, k^{\nu}_{\mu}) \equiv \mathcal{L}_{matt}(\varphi^{\alpha}, k^{\nu}_{\mu}\varphi^{\alpha}_{,\nu} - \mathcal{A}^{(a)}_{\mu}X^{\alpha}_{(a)\beta}\varphi^{\beta}), \qquad (23)
$$

although the expression $k^{\nu}_{\mu}\varphi^{\alpha}_{,\nu} - \mathcal{A}^{(a)}_{\mu}X^{\alpha}_{(a)\beta}\varphi^{\beta}$ can no longer be considered as a covariant derivative. Let us prove the gauge invariance of the action associated with this Lagrangian, i.e. let us see that

$$
\delta\hat{S}_{matt} = 0 \qquad (24)
$$

where

$$
\begin{aligned}
\hat{S}_{matt} &= \int \hat{L}_{matt}\, d^4x \equiv \int \Lambda\widehat{\mathcal{L}}_{matt}(\varphi^{\alpha}, \varphi^{\alpha}_{,\mu}, \mathcal{A}^{(a)}_{\nu}, k^{\nu}_{\mu})\, d^4x\\
&= \int \Lambda\mathcal{L}_{matt}(\varphi^{\alpha}, k^{\nu}_{\mu}\varphi^{\alpha}_{,\nu} - \mathcal{A}^{(a)}_{\mu}X^{\alpha}_{(a)\beta}\varphi^{\beta})\, d^4x, &(25)
\end{aligned}
$$

the factor Λ being a function of the tetrad fields to be determined by demanding the gauge invariance of $\hat{S}_{matt}$. The infinitesimal variation of $\widehat{\mathcal{L}}_{matt}(\varphi^{\alpha}, \varphi^{\alpha}_{,\mu}, \mathcal{A}^{(a)}_{\nu}, k^{\nu}_{\mu})$ under $G(M)$ reads:

$$
\begin{aligned}
\delta\widehat{\mathcal{L}}_{matt} &= f^{(a)}X^{\mu}_{(a)}\frac{\partial\widehat{\mathcal{L}}_{matt}}{\partial x^{\mu}} + f^{(a)}X^{\alpha}_{(a)\beta}\varphi^{\beta}\frac{\partial\widehat{\mathcal{L}}_{matt}}{\partial\varphi^{\alpha}} \\
&+ (\partial_{\mu}f^{(a)}X^{\alpha}_{(a)\beta}\varphi^{\beta} - \varphi^{\alpha}_{,\nu}(\partial_{\mu}f^{(a)}X^{\nu}_{(a)} + f^{(a)}\partial_{\mu}X^{\nu}_{(a)}) + f^{(a)}X^{\alpha}_{(a)\beta}\varphi^{\beta}_{,\mu})\frac{\partial\widehat{\mathcal{L}}_{matt}}{\partial\varphi^{\alpha}_{,\mu}} \\
&+ (f^{(b)}C^{a}_{bc}\mathcal{A}^{(c)}_{\mu} + k^{\nu}_{\mu}\partial_{\nu}f^{(a)} - f^{(b)}\mathcal{A}^{(a)}_{\sigma}\partial_{\mu}X^{\sigma}_{(b)})\frac{\partial\widehat{\mathcal{L}}_{matt}}{\partial\mathcal{A}^{(a)}_{\mu}} \\
&+ (X^{\nu}_{(a)}k^{\sigma}_{\mu}\partial_{\sigma}f^{(a)} + f^{(a)}(k^{\sigma}_{\mu}\partial_{\sigma}X^{\nu}_{(a)} - k^{\nu}_{\sigma}\partial_{\mu}X^{\sigma}_{(a)}))\frac{\partial\widehat{\mathcal{L}}_{matt}}{\partial k^{\nu}_{\mu}}.
\end{aligned} \tag{26}
$$

Let us consider the following change of variables:

$$
\begin{aligned}
\phi^{\alpha} &= \varphi^{\alpha} \\
\phi^{\alpha}_{,\mu} &= k^{\nu}_{\mu}\varphi^{\alpha}_{,\nu} - \mathcal{A}^{(a)}_{\mu}X^{\alpha}_{(a)\beta}\varphi^{\beta} \\
\mathcal{B}^{(a)}_{\mu} &= \mathcal{A}^{(a)}_{\mu} \\
K^{\mu}_{\nu} &= k^{\mu}_{\nu} \\
Q^{\mu}_{\nu} &= q^{\mu}_{\nu},
\end{aligned} \tag{27}
$$

and the corresponding change in the partial derivatives:

$$
\begin{aligned}
\frac{\partial}{\partial\varphi^{\alpha}} &= \frac{\partial}{\partial\phi^{\alpha}} - \mathcal{B}^{(a)}_{\mu}X^{\beta}_{(a)\alpha}\frac{\partial}{\partial\phi^{\beta}_{,\mu}} \\
\frac{\partial}{\partial\varphi^{\alpha}_{,\mu}} &= K^{\mu}_{\nu}\frac{\partial}{\partial\phi^{\alpha}_{,\nu}} \\
\frac{\partial}{\partial\mathcal{A}^{(a)}_{\mu}} &= \frac{\partial}{\partial\mathcal{B}^{(a)}_{\mu}} - X^{\alpha}_{(a)\beta}\phi^{\beta}\frac{\partial}{\partial\phi^{\alpha}_{,\mu}} \\
\frac{\partial}{\partial k^{\nu}_{\mu}} &= \frac{\partial}{\partial K^{\nu}_{\mu}} + Q^{\sigma}_{\nu}(\phi^{\alpha}_{,\sigma} + \mathcal{B}^{(a)}_{\sigma}X^{\alpha}_{(a)\beta}\phi^{\beta})\frac{\partial}{\partial\phi^{\alpha}_{,\mu}}.
\end{aligned} \tag{28}
$$

With the help of this change of variables the infinitesimal variation (26) under the local space-time symmetry group can be written as $f^{(a)}$ times the global variation of the original matter Lagrangian density of the theory depending on the field variables ϕ^{α} and $\phi^{\alpha}_{,\mu}$, i.e.

$$
\delta\widehat{\mathcal{L}}_{matt}(\varphi^{\alpha}, \varphi^{\alpha}_{,\mu}, \mathcal{A}^{(a)}_{\nu}, k^{\nu}_{\mu}) = f^{(a)}\delta^{global}_{(a)}\mathcal{L}_{matt}(\phi^{\alpha}, \phi^{\alpha}_{,\mu}) \tag{29}
$$

where

$$\delta^{global}_{(a)}\mathcal{L}_{matt}(\phi^\alpha,\phi^\alpha_{,\mu}) \equiv X^\nu_{(a)}\frac{\partial\mathcal{L}_{matt}}{\partial x^\nu} + X^\gamma_{(a)\beta}\phi^\beta\frac{\partial\mathcal{L}_{matt}}{\partial\phi^\gamma} + (X^\gamma_{(a)\beta}\phi^\beta_{,\nu} - \phi^\gamma_{,\sigma}\frac{\partial X^\sigma_{(a)}}{\partial x^\nu})\frac{\partial\mathcal{L}_{matt}}{\partial\phi^\alpha_{,\nu}}. \quad (30)$$

Using the hypothesis of invariance of the matter action under the global group (see (16)) it follows that

$$\begin{aligned}\delta\widehat{\mathcal{L}}_{matt}(\varphi^\alpha,\varphi^\alpha_{,\mu},\mathcal{A}^{(a)}_\nu,k^\nu_\mu) &= -f^{(a)}\mathcal{L}_{matt}(\phi^\alpha,\phi^\alpha_{,\mu})\partial_\mu X^\mu_{(a)} \\ &= -f^{(a)}\mathcal{L}_{matt}(\varphi^\alpha,k^\nu_\mu\varphi^\alpha_{,\nu}-\mathcal{A}^{(a)}_\mu X^\alpha_{(a)\beta}\varphi^\beta)\partial_\mu X^\mu_{(a)}. \end{aligned} \quad (31)$$

Let us determine the simplest form of the factor Λ that leads to a gauge invariant Lagrangian density $\widehat{L}_{matt} \equiv \Lambda\widehat{\mathcal{L}}_{matt}$. $\widehat{L}_{matt}$ must satisfy the condition:

$$\delta\widehat{L}_{matt} + \widehat{L}_{matt}\partial_\mu(f^{(a)}X^\mu_{(a)}) = 0. \quad (32)$$

More explicitly,

$$\delta\Lambda\widehat{\mathcal{L}}_{matt} + \Lambda\delta\widehat{\mathcal{L}}_{matt} + \Lambda\widehat{\mathcal{L}}_{matt}\partial_\mu f^{(a)}X^\mu_{(a)} + \Lambda\widehat{\mathcal{L}}_{matt}f^{(a)}\partial_\mu X^\mu_{(a)} = 0\,. \quad (33)$$

Assuming that

$$\widehat{\mathcal{L}}_{matt}(\varphi^\alpha,\varphi^\alpha_{,\mu},\mathcal{A}^{(a)}_\nu,k^\nu_\mu) = \mathcal{L}_{matt}(\varphi^\alpha,k^\nu_\mu\varphi^\alpha_{,\nu}-\mathcal{A}^{(a)}_\mu X^\alpha_{(a)\beta}\varphi^\beta) \quad (34)$$

and using (31), the gauge invariance condition of $\widehat{L}_{matt}$ (32) provides the equation that Λ must satisfy, that is:

$$\delta\Lambda + \Lambda\partial_\mu f^{(a)}X^\mu_{(a)} = 0. \quad (35)$$

For simplicity we shall assume that Λ only depends on the tetrad fields, so that

$$\delta\Lambda = \frac{\partial\Lambda}{\partial k^\nu_\mu}\delta k^\nu_\mu\,, \quad (36)$$

and taking into account (20) the final equation that determines the form of Λ reads:

$$(X^\nu_{(a)}k^\sigma_\mu\partial_\sigma f^{(a)} + f^{(a)}(k^\sigma_\mu\partial_\sigma X^\nu_{(a)} - k^\nu_\sigma\partial_\mu X^\sigma_{(a)}))\frac{\partial\Lambda}{\partial k^\nu_\mu} + \Lambda\partial_\mu f^{(a)}X^\mu_{(a)} = 0. \quad (37)$$

Since the functions $f^{(a)}$ are arbitrary and independent, the coefficients of $f^{(a)}$ and their first-order derivatives must be zero, so that we obtain the following system of partial differential equations:

$$a) \qquad f^{(a)} : (k^\sigma_\mu \partial_\sigma X^\nu_{(a)} - k^\nu_\sigma \partial_\mu X^\sigma_{(a)}) \frac{\partial \Lambda}{\partial k^\nu_\mu} = 0 \tag{38}$$

$$b) \qquad \partial_\sigma f^{(a)} : X^\nu_{(a)} k^\sigma_\mu \frac{\partial \Lambda}{\partial k^\nu_\mu} + \Lambda X^\sigma_{(a)} = 0, \tag{39}$$

and the general solution for this system is

$$\Lambda = det(q^\nu_\mu). \tag{40}$$

Note that when $k^\nu_\mu \to \delta^\nu_\mu$ (internal symmetry case) then $\Lambda \to 1$. As a corollary, we can assert that *the new action invariant under the local space-time symmetry describing the matter fields as well as their interaction with the compensating (gauge) fields* $\mathcal{A}^{(a)}_\nu$, k^ν_μ *reads*

$$\hat{S}_{matt} = \int \hat{L}_{matt} \, d^4x \equiv \int \Lambda \widehat{\mathcal{L}}_{matt} \, d^4x, \tag{41}$$

where $\Lambda \equiv det(q^\nu_\mu)$.

If we introduce new "tetrad-like" compensating fields $h^{(a)\nu}_{\mu\sigma}$ associated with each generator by means of the decomposition of the tetrad field

$$k^\nu_\mu = \delta^\nu_\mu - h^{(a)\nu}_{\mu\sigma} X^\sigma_{(a)} \tag{42}$$

we can write the interaction term in the way

$$\varphi^\alpha_{,\mu} - \mathcal{A}^{(a)}_\mu X^\alpha_{(a)\beta} \varphi^\beta - h^{(a)\nu}_{\mu\sigma} X^\sigma_{(a)} \varphi^\alpha_{,\nu} \tag{43}$$

that generalizes more directly the case of internal symmetry (a similar expression was already suggested in a footnote in [6]). ¿From this expression we can observe that while the gauge potentials associated with the internal action of the group couple to the matter fields, the fields $h^{(a)\nu}_{\mu\sigma}$ couple to the derivatives of the matter fields. Note also that the two indices of k^ν_μ transform according to different transformation rules, i.e. while the index ν transforms as a tensor, the index μ inherits the non-tensorial character of $h^{(a)\nu}_{\mu\sigma}$. We shall not make an explicit distinction in the notation for the tetrad indices. No confusion should arise since tetrads (k) and their inverse (q) are denoted differently.

As far as the Lagrangian $\mathcal{L}_0$ for the free compensating fields is concerned we can establish the following theorem: *The necessary condition for* $\mathcal{L}_0$ *to be invariant under the current group* $G(M)$ *is that* $\mathcal{L}_0$ *depends on the fields* $\mathcal{A}^{(a)}_\mu$, k^ν_μ *and their "derivatives"* $\mathcal{A}^{(a)}_{\mu,\nu}$, $k^\nu_{\mu,\sigma}$ *only through the specific combination (generalized "curvature"):*

$$\mathcal{F}^{(a)}_{\mu\nu} \equiv \mathcal{A}^{(a)}_{\mu,\sigma} k^\sigma_\nu - \mathcal{A}^{(a)}_{\nu,\sigma} k^\sigma_\mu - \frac{1}{2} C^a_{bc} (\mathcal{A}^{(b)}_\mu \mathcal{A}^{(c)}_\nu - \mathcal{A}^{(b)}_\nu \mathcal{A}^{(c)}_\mu) - \mathcal{A}^{(a)}_\sigma T^\sigma_{\mu\nu} \, ,$$

with $T^\sigma_{\mu\nu} \equiv q^\sigma_\rho (k^\rho_{\mu,\tau} k^\tau_\nu - k^\rho_{\nu,\tau} k^\tau_\mu)$.

The gauge-invariant action for the compensating fields has the form

$$\widehat{S}_0 = \int \widehat{L}_0 \, d^4x \equiv \int \Lambda \widehat{\mathcal{L}}_0 \, d^4x. \tag{44}$$

Now that a generalized gauge theory including space-time symmetries is available, different gauge gravitational theories can be constructed using several space-time symmetry groups: space-time translations, Lorentz group, Poincaré group, Weyl group, etc. and the resulting theories can be reduced to the Einstein's theory in some particular cases. As an example, and since we shall be concerned with the Poincaré group in the next section, the rest of this subsection will be devoted to the gravitational theory associated with the gauge theory of the Poincaré group (see [6] among others).

The notation for the Poincaré group (semidirect product of the translations group and Lorentz group) index is $(a) = \{(\mu)$ translations, $(\nu\sigma)$ Lorentz$\}$ and a particular realization for the generators of the rigid Poincaré algebra reads:

Translations:

$$\delta_{(\mu)} x^\nu = \delta^\nu_\mu \tag{45}$$
$$\delta_{(\mu)} \varphi^\alpha = 0 \tag{46}$$

Lorentz:

$$\delta_{(\mu\nu)} x^\sigma = \delta^\sigma_{(\mu\nu),\rho} x^\rho \equiv (\delta^\sigma_\mu \eta_{\nu\rho} - \delta^\sigma_\nu \eta_{\mu\rho}) x^\rho \tag{47}$$
$$\delta_{(\mu\nu)} \varphi^\alpha = S^\alpha_{(\mu\nu)\beta} \varphi^\beta \tag{48}$$

and the form of $S^\alpha_{(\mu\nu)\beta}$ is determined by the commutation relations of the Poincaré group and antisymmetry in the Lorentz indices $S^\alpha_{(\mu\nu)\beta} = -S^\alpha_{(\nu\mu)\beta}$.

In the present case, the Lagrangian for the free compensating fields $\mathcal{A}^{(\nu)}_\mu$, $\mathcal{A}^{(\nu\sigma)}_\mu$, k^ν_μ is an arbitrary function of the translational and Lorentz generalized curvatures, according to the previous general theory of gauged space-time algebras,

$$\mathcal{L}_0 = \mathcal{L}_0(\mathcal{F}^{(\mu)}_{\nu\sigma}, \mathcal{F}^{(\nu\sigma)}_{\rho\theta}). \tag{49}$$

As a particular case we can choose

$$\mathcal{L}_0 = \mathcal{L}_0(\mathcal{F}^{(\nu\sigma)}_{\rho\theta}). \tag{50}$$

and by means of the decomposition of the tetrad fields in terms of the translational gauge fields

$$k^\mu_\nu = \delta^\mu_\nu + \mathcal{A}^{(\mu)}_\nu \tag{51}$$

one can obtain (by combining the equations of motion associated with $\mathcal{A}_{\nu}^{(\mu)}$ and k_{μ}^{ν}) the following generalized Einstein's equation:

$$\mathcal{F}_{\mu\nu}^{(\sigma\rho)}\frac{\partial L_0}{\partial \mathcal{F}_{\epsilon\nu}^{(\sigma\rho)}}-\frac{1}{2}\delta_{\mu}^{\epsilon}L_0=-\frac{1}{2}k_{\xi}^{\epsilon}t_{\mu}^{\xi}\,, \tag{52}$$

where

$$L_0 \equiv \Lambda\mathcal{L}_0(\mathcal{F}_{\rho\theta}^{(\nu\sigma)}), \tag{53}$$

$$t_{\nu}^{\mu} \equiv q_{\sigma}^{\mu}\left(-\delta_{\nu}^{\sigma}\widehat{L}_{matt}+\frac{\partial\widehat{L}_{matt}}{\partial\varphi_{,\sigma}^{\alpha}}\phi_{\rho}^{\alpha}q_{\nu}^{\rho}\right), \tag{54}$$

$$\phi_{\rho}^{\alpha} \equiv k_{\rho}^{\nu}\varphi_{,\nu}^{\alpha}-\mathcal{A}_{\rho}^{(a)}X_{(a)\beta}^{\alpha}\varphi^{\beta}, \tag{55}$$

$$\widehat{L}_{matt} = \Lambda\mathcal{L}_{matt}(\varphi^{\alpha},\phi_{\mu}^{\alpha}). \tag{56}$$

Let us consider two cases:

A) Equations in vacuum: The action reduces to that for the free compensating fields,

$$\mathcal{S}_0=\int(\Lambda\mathcal{L}_0)d^4x\,, \tag{57}$$

and, with the choice $\mathcal{L}_0=\mathcal{F}_{\sigma\rho}^{(\mu\nu)}\eta_{\mu}^{\sigma}\eta_{\nu}^{\rho}$, the equation of motion $\frac{\delta L_{tot}}{\delta\mathcal{A}_{\mu}^{(a)}}=0$ yields

$$\mathcal{A}_{(\sigma\rho)\mu}=\frac{1}{2}T_{\mu\sigma\rho}+\frac{1}{2}(T_{\sigma\rho\mu}-T_{\rho\sigma\mu})\,, \tag{58}$$

where $\mathcal{A}_{(\sigma\rho)\mu}\equiv\mathcal{A}_{\mu}^{(\theta\epsilon)}\eta_{\theta\sigma}\eta_{\epsilon\rho}$ and $T_{\mu\sigma\rho}\equiv T_{\sigma\rho}^{\nu}\eta_{\nu\mu}$. Note that $\mathcal{A}_{\mu}^{(\sigma\rho)}=-\mathcal{A}_{\mu}^{(\rho\sigma)}$ and $T_{\sigma\rho}^{\nu}=-T_{\rho\sigma}^{\nu}$. Substituting (58) into L_0, one easily finds that the theory reduces to Einstein's vacuum theory.

B) Equations with matter: In this case the total action must include a matter piece which should be made explicit. Then, only general comments can be pointed out. For instance, the expression (58) now reads

$$\mathcal{A}_{(\sigma\rho)\mu}=\frac{1}{2}T_{\mu\sigma\rho}+\frac{1}{2}(T_{\sigma\rho\mu}-T_{\rho\sigma\mu})+M_{(\sigma\rho)\mu}\,, \tag{59}$$

where the extra term $M_{(\sigma\rho)\mu}$ is zero for spinless matter but not for fermionic matter. Then, for a Dirac spinor ψ, $M_{(\sigma\rho)\mu}$ is proportional to $\bar{\psi}\gamma_{\mu}\Sigma_{\sigma\rho}\psi$ and this term is known as the *contortion* created by spinors [11]. See also [6].

This situation generalizes Einstein's theory with a Lagrangian density,

$$\Lambda\mathcal{F}_{\mu\nu}^{(\mu\nu)}=\Lambda R^{(\Gamma_{Levi-Civita})}+\Upsilon(M_{(\sigma\rho)\mu})\,, \tag{60}$$

where the form of the function Υ again depends on the specific nature of fermionic matter.

Had $\mathcal{L}_0$ depended also on $\mathcal{F}^{(\mu)}_{\nu\sigma}$ we would have obtained a theory even more general than Einstein's, known as Einstein-Cartan theory, in which $\mathcal{F}^{(\mu\nu)}_{\sigma\rho}$ is interpreted as a curvature and $\mathcal{F}^{(\mu)}_{\nu\sigma}$ as a torsion.

We would like to remark that the formulation of the gauge theory associated with space-time symmetry groups, which has been presented in this subsection, not only can be applied to groups higher than the Poincaré group (in this sense this theory would be more general than that of [6]) as for instance the Weyl group, but also, this framework results specially suitable for the unification of interactions. The crucial point is the incorporation of non-trivial gauge translational potentials even though the corresponding generators do not act on the internal components of the matter fields.

3 Towards a mixing of gravity and electromagnetism

The present section is devoted to a simple, yet non-trivial framework to account for the mixing of gravitation and the rest of fundamental interactions. In our approach we make use of two important physical notions: the well-known gauge invariance principle and the concept of central extension of a group (in particular, the central extension of Poincaré group, $\mathcal{P}$, by $U(1)$, denoted in the following as $\tilde{\mathcal{P}}$). On the one hand, the gauge invariance is the key for the understanding of the formulation of the interactions and is a requirement that helps to achieve renormalizability. Moreover, the interest in the description of gravity as a gauge theory is precisely the possibility of its unification with the rest of interactions. On the other hand, the motivation for considering a centrally extended group is based on the relevance of this notion in some areas of physics, specially in quantum theory (also in classical mechanics in the Hamilton-Jacobi approach). In fact, traditional space-time groups as Galilei or Poincaré groups leave only semi-invariant the Lagrangians of the corresponding free particles, and a central extension is required to achieve strict invariance. It is also well known that the Schrödinger equation for the free particle is not invariant under the Galilei group G although it is under the centrally extended Galilei group $\tilde{G}_{(m)}$[2]. Analogously, we can consider the space-time symmetry of the quantum relativistic particle, which is characterized by the commutator of boosts and translations modified with the central generator Ξ associated with $U(1)$, i.e.

$$[K^i, P_j] = \delta^i_j \left(\frac{1}{c}P_0 + \lambda^0 \Xi\right) . \tag{61}$$

with $\lambda^0 \equiv m$. In this case, a particular four-vector λ of the orbit $\lambda^2 = m^2$ in the momentum space has been chosen. In the non-relativistic limit this commutator yields the basic commutators of the centrally extended Galilei group, $\widetilde{G}_{(m)}$.

[2]The particular case of central extensions of Lie groups by $U(1)$ (whose classification was carried out long ago by Bargmann [41]) is very important from the physical point of view. In fact, it is known that the question of the classification of all the possible projective unitary representations of a group (which are the relevant representations in quantum mechanics) is equivalent to the problem of the classification of the central extensions of a group by $U(1)$.

In the present paper we shall approach the mixing between electromagnetism and gravity by studying the gauge symmetry of the central extension of Poincaré group by $U(1)$, denoted by $\widetilde{\mathcal{P}}$. The group index (a) now runs over $\{(\mu)$ translation, $(\nu\sigma)$ Lorentz, $(\Phi)\ U(1)\}$. The commutator of Lorentz and translations generators is modified according to

$$[\widetilde{M}_{\mu\nu}, \widetilde{P}_\rho] = \eta_{\nu\rho}\widetilde{P}_\mu - \eta_{\mu\rho}\widetilde{P}_\nu - (\lambda_\mu\eta_{\nu\rho} - \lambda_\nu\eta_{\mu\rho})\Xi \equiv C^\sigma_{\mu\nu,\rho}\widetilde{P}_\sigma + C^\Phi_{\mu\nu,\rho}\Xi, \tag{62}$$

with

$$C^\Phi_{\mu\nu,\rho} \equiv \lambda_\nu\eta_{\mu\rho} - \lambda_\mu\eta_{\nu\rho}, \tag{63}$$

where Ξ is the generator of $U(1)$ and λ_μ is a vector in the Poincaré coalgebra belonging to a given coadjoint orbit, and will be related later to the coupling constant of the mixing. From the strict mathematical point of view, the group $\widetilde{\mathcal{P}}$ is a trivial central extension of the Poincaré group by $U(1)$. In fact, by making the replacement

$$P_\mu \rightarrow \widetilde{P}_\mu = P_\mu + \lambda_\mu\Xi \tag{64}$$

it becomes clear that $\widetilde{\mathcal{P}}$ is equivalent to $\mathcal{P} \otimes U(1)$. Therefore the associated co-cycle is trivial, i.e. co-boundary. It is known, however, that trivial co-cycles can be divided into two different types depending on the structure of their generating functions [42]. The first type comprises the co-boundaries which are really physically trivial as they lead to zero curvature. The second type (and the truly relevant from the physical point of view) corresponds to those co-boundaries leading to a group connection with non-trivial curvature and are called pseudo-co-cycles. The central extensions that they provide are referred to as pseudo-extensions. The most remarkable fact is that non-trivial symplectic structures and dynamics can be derived out of them [43, 44]. An example of pseudo-extension is the case of the central extension of the Poincaré group by $U(1)$. As we shall see in the present section, this group is associated with a gauge symmetry which in particular generates a $U(1)$-field strength (containing terms of pure gravitational origin) that is not trivial as a consequence of the associated co-boundary being a pseudo-co-cycle.[3]

Let us consider the gauge theory of $\widetilde{\mathcal{P}}$. Proceeding according to the general theory developed in the subsection 2.2, the Lagrangian for the free compensating fields should be a general function of the generalized curvatures:

[3] The characterization of the classes of pseudo-extensions associated with non-equivalent symplectic structures leads to the notion of pseudo-cohomology. As a report on pseudo-extensions, and the role that they play in representation theory, we refer the reader to Ref. [45] and references there in. Here we would like to mention briefly some indications of the need of pseudo-cohomology. It is known that pseudo-co-cycles play a fundamental role in representation theory of semi-simple groups (including infinite-dimensional ones like $Diff(S^1)$ and other diffeomorphism groups) and also in the explicit construction of the local exponent associated with Lie algebra co-cycles of the corresponding Kac-Moody groups [44, 46]. In any case, the framework where the need and relevance of pseudo-cohomology is more patent is the so-called Group Approach to Quantization (GAQ) (mentioned in the Introduction).

$$\mathcal{L}_0 = \mathcal{L}_0(\mathcal{F}^{(\sigma)}_{\mu\nu},\, \mathcal{F}^{(\sigma\rho)}_{\mu\nu},\, \mathcal{F}^{(\Phi)}_{\mu\nu})\,. \tag{65}$$

Let us define the fields $A^{(a)}_\mu \equiv q^\nu_\mu \mathcal{A}^{(a)}_\nu$ and write the curvatures in the following way:

$$\mathcal{F}^{(a)}_{\mu\nu} \equiv k^\sigma_\mu k^\rho_\nu F^{(a)}_{\sigma\rho}\,, \tag{66}$$

where

$$F^{(a)}_{\sigma\rho} \equiv A^{(a)}_{\sigma,\rho} - A^{(a)}_{\rho,\sigma} + \frac{1}{2}\widetilde{C^a_{bc}}(A^{(b)}_\sigma A^{(c)}_\rho - A^{(b)}_\rho A^{(c)}_\sigma)\,. \tag{67}$$

Here, (a) runs over the entire group $\widetilde{\mathcal{P}}$ and $\widetilde{C^a_{bc}}$ denotes its structure constants. The presence of a coupling constant of the mixing, κ, through $C^\Phi_{\mu,\sigma\rho}$ in the generalized curvature $\mathcal{F}^{(\Phi)}_{\mu\nu}$, due to the central pseudo-extension, is to be remarked. In fact, and without loss of generality we can select a preferred direction for λ_μ,

$$\lambda_\mu = -\kappa\delta^0_\mu, \tag{68}$$

so that we arrive at

$$C^\Phi_{\mu,\sigma\rho} \equiv -\kappa(\eta_{\rho\mu}\delta^0_\sigma - \eta_{\sigma\mu}\delta^0_\rho). \tag{69}$$

In the context of the gauge theory of Poincaré group the Lorentz curvature is enough to recover Einstein gravity in vacuum, as was pointed out in the subsection 2.2. Therefore, in the present model it is enough to consider only the Lorentz and $U(1)$ generalized curvatures in order to construct an electro-gravity theory in the most economical way. The expression of such curvatures reads respectively:

$$\begin{aligned}
F^{(\epsilon\rho)}_{\mu\nu} &= A^{(\epsilon\rho)}_{\mu,\nu} - A^{(\epsilon\rho)}_{\nu,\mu} - \eta_{\theta\sigma}(A^{(\epsilon\theta)}_\mu A^{(\sigma\rho)}_\nu - A^{(\epsilon\theta)}_\nu A^{(\sigma\rho)}_\mu)\,,\\
F^{(\Phi)}_{\mu\nu} &= A^{(\Phi)}_{\mu,\nu} - A^{(\Phi)}_{\nu,\mu} - \frac{1}{2}C^\Phi_{\epsilon,\theta\rho}(A^{(\epsilon)}_\mu A^{(\theta\rho)}_\nu - A^{(\epsilon)}_\nu A^{(\theta\rho)}_\mu)\\
&= A^{(\Phi)}_{\mu,\nu} - A^{(\Phi)}_{\nu,\mu} + \kappa\eta_{ij}(A^{(j)}_\mu A^{(0i)}_\nu - A^{(j)}_\nu A^{(0i)}_\mu)\,,
\end{aligned} \tag{70}$$

where η_{ij} is the Minkowski metric tensor and the latin indices i, j run from 1 to 3 and we recall that $A^{(\epsilon)}_\theta \equiv q^\nu_\theta \mathcal{A}^{(\epsilon)}_\nu = q^\nu_\theta(k^\epsilon_\nu - \delta^\epsilon_\nu) = \delta^\epsilon_\theta - q^\epsilon_\theta$.

The standard Einstein-Maxwell theory can be described by the gauge theory associated with the direct product of the Poincaré and $U(1)$ groups. But in our present approach corresponding to the central extension the $U(1)$ gauge potential is no longer the usual electromagnetic field $A^{(elec)}_\mu$ in the presence of a gravitational field; rather $A^{(\Phi)}_\mu$ must contain it at zero order in the coupling constant κ to account for the limit of the theory without mixing, i.e.

$$A^{(\Phi)}_\mu = A^{(elec)}_\mu + \kappa B^{(grav)}_\mu\,. \tag{71}$$

In this expression $B^{(grav)}_\mu$ is an "electromagnetic" contribution of pure gravitational origin (note that $B^{(grav)}_\mu$ must be a function of the gravitational potentials). The theory can be developed working up to first order in κ and this is, in fact, a good approximation to the problem due to the small value of the coupling constant κ (it should be expected that $|\kappa q| \leq m_{electron}$ and therefore κ would result to be $\leq 6 \times 10^{-12}$ Kg/C [38])[4].

Hence, the curvature associated with $U(1)$ can be decomposed into two pieces: the usual electromagnetic curvature in a gravitational background $F^{(elec)}_{\mu\nu}$ added to a contribution constructed from the gravitational potentials $F^{(grav)}_{\mu\nu}$, i. e.

$$F^{(\Phi)}_{\mu\nu} = F^{(elec)}_{\mu\nu} + \kappa F^{(grav)}_{\mu\nu} \tag{72}$$

with

$$\begin{aligned} F^{(elec)}_{\mu\nu} &= A^{(elec)}_{\mu,\nu} - A^{(elec)}_{\nu,\mu} \,, \\ F^{(grav)}_{\mu\nu} &= B^{(grav)}_{\mu,\nu} - B^{(grav)}_{\nu,\mu} + \eta_{ij}(A^{(j)}_\mu A^{(0i)}_\nu - A^{(j)}_\nu A^{(0i)}_\mu) \,. \end{aligned} \tag{73}$$

As a result we propose that the field $B^{(grav)}_\mu$ could be responsible for some electromagnetic force associated with very massive rotating systems, as $A^{(0i)}_\mu$ is somehow related to "Coriolis-like forces"[5].

The simplest electro-gravitational gauge invariant Lagrangian density for the free compensating fields in our model has the form:

$$\begin{aligned} L_0 &\sim \Lambda(\mathcal{F}^{(\Phi)}_{\mu\nu}\mathcal{F}^{(\Phi)\mu\nu} + \mathcal{F}^{(\mu\nu)}_{\mu\nu}) \\ &= \Lambda(g^{\mu\sigma}g^{\nu\rho}F^{(\Phi)}_{\mu\nu}F^{(\Phi)}_{\sigma\rho} + k^\sigma_\mu k^\rho_\nu F^{(\mu\nu)}_{\sigma\rho}), \end{aligned} \tag{74}$$

where $\mathcal{F}^{(\Phi)\mu\nu} \equiv \mathcal{F}^{(\Phi)}_{\mu\nu}\eta^{\sigma\mu}\eta^{\rho\nu}$, $g^{\sigma\rho} = k^\sigma_\mu k^\rho_\nu \eta^{\mu\nu}$ and $\Lambda = det(q^\nu_\mu)$.

The Euler-Lagrange motion equations read:

$$(1): \quad \frac{\partial L_0}{\partial A^{(\nu\rho)}_\mu} - \frac{\partial}{\partial x^\sigma}\left(\frac{\partial L_0}{\partial A^{(\nu\rho)}_{\mu,\sigma}}\right) = 0 \;\; \Rightarrow \tag{75}$$

$$C^\Phi_{\sigma,\epsilon\theta}A^{(\sigma)}_\nu F^{(\Phi)\mu\nu} + k^\mu_\rho T^\rho_{\epsilon\theta} - k^\mu_\theta T^\rho_{\epsilon\rho} + k^\mu_\epsilon T^\rho_{\theta\rho} + (k^\mu_\rho k^\nu_\theta - k^\mu_\theta k^\nu_\rho)A^{(\rho}{}_{\epsilon)\nu} - (k^\mu_\epsilon k^\nu_\rho + k^\mu_\rho k^\nu_\epsilon)A^{(\rho}{}_{\theta)\nu} = 0\,,$$

[4] The maximum supposed value for κ would correspond to the mass-charge relation of the electron. In this case, the physical content of the module of λ_μ would be essentially the quotient of coupling constants (gravitational and electromagnetic ones). This is in fact a feature of unified (gauge) theories, for example, in the electro-weak theory the tangent of the Weinberg angle gives precisely the relation between the isospin and hypercharge coupling constants.

[5] Note that the Lorentz potentials $A^{(0i)}_\mu$ can be related to the components $\Gamma^i_{00}, \Gamma^i_{0k}, \Gamma^i_{jk}$ of the Christoffel symbols which produce a Coriolis-like force on a particle in a constant gravitational field [47].

where $F^{(\Phi)\mu\nu} = F^{(\Phi)}_{\rho\lambda} g^{\rho\mu} g^{\lambda\nu}$, $g^{\rho\mu} = k^{\rho}_{\sigma} k^{\mu}_{\theta} \eta^{\sigma\theta}$, $A^{(\mu}{}_{\nu)\sigma} \equiv \eta_{\nu\rho} A^{(\mu\rho)}_{\sigma}$ and $T^{\rho}_{\epsilon\theta} = q^{\rho}_{\mu}(k^{\mu}_{\epsilon,\tau} k^{\tau}_{\theta} - k^{\mu}_{\theta,\tau} k^{\tau}_{\epsilon})$;

$$(2): \qquad \frac{\partial L_0}{\partial A^{(\Phi)}_{\mu}} - \frac{\partial}{\partial x^{\sigma}} \left(\frac{\partial L_0}{\partial A^{(\Phi)}_{\mu,\sigma}} \right) = 0 \quad \Rightarrow \tag{76}$$

$$\frac{d}{dx^{\sigma}} (\Lambda F^{(\Phi)\mu\sigma}) = 0 \; ;$$

$$(3): \qquad \frac{\partial L_0}{\partial k^{\mu}_{\nu}} - \frac{\partial}{\partial x^{\sigma}} \left(\frac{\partial L_0}{\partial k^{\mu}_{\nu,\sigma}} \right) = 0 \quad \Rightarrow \tag{77}$$

$$F^{(\nu\sigma)}_{\mu\sigma} - \frac{1}{2} \delta^{\nu}_{\mu} F^{(\sigma\lambda)}_{\sigma\lambda} = \mathcal{T}^{\nu}_{\mu} \; ,$$

where

$$\mathcal{T}^{\nu}_{\mu} \equiv \mathcal{T}^{\nu(mix)}_{\mu} + \mathcal{T}^{\nu(\Phi)}_{\mu} \, .$$

The tensor $\mathcal{T}^{\nu(\Phi)}_{\mu} \equiv -F^{(\Phi)\nu}_{\sigma} F^{(\Phi)\sigma}_{\mu} + \frac{1}{2} \delta^{\nu}_{\mu} F^{(\Phi)}_{\sigma\lambda} F^{(\Phi)\sigma\lambda}$, generalizes the energy-momentum tensor corresponding to the electromagnetic field in a gravitational field (with $F^{(\Phi)\nu}_{\sigma} = g^{\lambda\nu} F^{(\Phi)}_{\sigma\lambda}$) and the piece $\mathcal{T}^{\nu(mix)}_{\mu} = \frac{1}{2} C^{\Phi}_{\mu,\theta\epsilon} q^{\nu}_{\rho} F^{(\Phi)\rho\tau} A^{(\theta\epsilon)}_{\tau}$ is completely new and arises as a direct consequence of the mixing of the space-time and internal symmetries.

In order to proceed further in the understanding of the proposed model we shall consider the effects of the mixing of gravity and electromagnetism in the "geodesic" motion. Let us consider a spinless particle of mass m, momentum $p_{\mu} (= m u_{\mu} = m \frac{dx_{\mu}}{d\tau})$ and charge e. According to the (Generalized) Minimal Coupling Principle, the Lagrangian of the free particle

$$\mathcal{L}_{particle} = \frac{1}{2m} p_{\mu} p_{\nu} \eta^{\mu\nu} \tag{78}$$

must be replaced by the modified Lagrangian where $p_{\mu} \to k^{\nu}_{\mu}(p_{\nu} - e A^{(\Phi)}_{\nu}) = k^{\nu}_{\mu}(p_{\nu} - e A^{(elec)}_{\nu} - \kappa e B^{(grav)}_{\nu})$:

$$\widehat{\mathcal{L}}_{particle} = \frac{1}{2} m u^{\mu} u^{\nu} g_{\mu\nu} - e u^{\mu} A^{(elec)\nu} g_{\mu\nu} - \kappa e u^{\mu} B^{(grav)\nu} g_{\mu\nu} \, , \tag{79}$$

where we have already neglected the misleading term $\frac{e^2}{2m} A^{(\Phi)\mu} A^{(\Phi)\nu} g_{\mu\nu}$ [6] , which, by the way, does not appear when working directly with the Poincaré-Cartan form instead of the

[6] We recall that already in the standard formulation of the Lorentz force in a gravitational field (without mixing) the interaction Lagrangian L_{int}, among some other requisites, must be linear in the charge of the particle and in the electromagnetic potential to account for the Lorentz invariance of γL_{int} (with $\gamma \equiv (1 - (\frac{u}{c})^2)^{-\frac{1}{2}}$) as a consequence of the requirement of Lorentz invariance of the action integral written in terms of the proper time τ [48].

Lagrangian. We also consider (74) as the Lagrangian density for the free compensating fields. As regards the interaction between a particle and a field, in general, it is required to distinguish between the coordinates y^σ where the fields are evaluated and the coordinates x^σ for the particle. In $\widehat{\mathcal{L}}_{particle}$ the fields are evaluated at the position of the particle, where the interaction occurs, but in $\mathcal{L}_0$ the fields are evaluated at y^σ.

The equation for the particle, $\frac{\partial \widehat{\mathcal{L}}_{particle}}{\partial x^\sigma} - \frac{d}{d\tau}\left(\frac{\partial \widehat{\mathcal{L}}_{particle}}{\partial u^\sigma}\right) = 0$ results in the usual motion equation for a particle in the presence of both gravitational and electromagnetic fields, with an additional Lorentz-like force (proportional to κe) generated by the gravitational potentials, i.e.

$$g_{\mu\sigma}\frac{du^\mu}{d\tau} = -u^\mu u^\nu \Gamma^{(L-C)}_{\mu\nu,\sigma} - \frac{e}{m}u^\mu F^{(elec)}_{\mu\sigma} - \frac{\kappa e}{m}u^\mu(\partial_\sigma B^{(grav)}_\mu - \partial_\mu B^{(grav)}_\sigma)\,. \tag{80}$$

with $\Gamma^{(L-C)}_{\mu\nu,\sigma} = \frac{1}{2}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$ being the Levi-Civita connection associated with the metric $g_{\mu\nu} = q^\rho_\mu q^\nu_\sigma \eta_{\rho\nu}$.
Considering the non-relativistic limit ($c \to \infty$) on the Poincaré group (stated as an Inonu-Wigner Lie algebra contraction [49]) the explicit form of the field $B^{(grav)}_\mu$ is then very simple ($h^i \equiv g^{0i} - \eta^{0i}$):

$$\begin{aligned} B^0 &= -\frac{\vec{h}^2}{8} \\ B^i &= -\frac{h^i}{2}\,. \end{aligned} \tag{81}$$

It must be remarked, however, that in any case we do not refer to a new force but, just, a mixing of interactions, so that the number of field degrees of freedom are the same that in the $\kappa \to 0$ limit.

Some final comments are in order: firstly, since the present theory has been formulated on symmetry grounds, it could be possible to attempt the quantization on the basis of the Group Approach to Quantization. With regard this question the purpose of the GAQ treatment for the quantization of gravity would consist in restricting ourselves to a subgroup of the supposed symmetry group of gravity. Thus using this subgroup to parametrize the corresponding solution submanifold (Schwarzschild-like solution, for instance) one could manage to describe the theory with a lower number of parameters (even finite) in a non-perturbative framework, then avoiding renormalizability problems. Secondly, the unification of gravity and electromagnetism here proposed can be inmediately generalized to the rest of interactions once the group $U(1)$ is considered as a subgroup of $(SU(2) \otimes U(1))/Z_2$, $SU(5)$ or any other "grand unification group". Finally, we also remark that the semidirect product of the diffeomorphism group of the space-time and the gauge group, $Diff(M) \otimes_S G(M)$, provides an extra natural mixing between gravity and the rest of (internal) interactions, although maybe less drastic in phenomenological terms than the mixing proposed here. In fact, in the case of electromagnetism, the semidirect action of the group of diffeomorphisms on the gauge group $U(1)(M)$ would account for diagrams in which photons and gravitons produce gravitons. Thus this mixing would result in a new modified dispersion relation between gravitons and photons. However in

the context of gauging the central extension of Poincaré group by $U(1)$, diagrams in which two gravitons provide one photon would enter the theory. In such a case the production of photons in the absence of electrically charged sources would be expected.

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References

[1] H. Weyl, Ann. Physik **59**, 101 (1919).

[2] H. Georgi and S.L. Glashow, Phys. Rev. Lett. **32**, 438 (1974).

[3] H. Georgi *in Particles and Fields 1974*, ed. C. Carlson, Amer. Inst. of Physics, New York (1975).

[4] S. Weinberg, *The Quantum Theory of Fields*, Vol.II, Cambridge U.P. (1996).

[5] R. Utiyama, Phys. Rev. **101**, 1597 (1956).

[6] T.W.B. Kibble, J. Math. Phys. **2**, 212 (1961).

[7] B.N. Frolov, Vestnik Moskovskogo Universiteta, Seria Fizika, astron. 6, 48, (1963) (in Russian).

[8] B.N. Frolov, Sovremennye problemy gravitatsii, Proc. of the 2nd Soviet Grav. Conf. 270, (1967) (in Russian)

[9] B.N. Frolov, Puankare Kalibrovochnaya teoriya gravitatsii, MGPU, Prometey, Moscow, (2003) (in Russian).

[10] Y.M. Cho, Phys. Rev. **D14**, 2521 (1976).

[11] Y.M. Cho, Phys. Rev. **D14**, 3335 (1976).

[12] D. Ivanenko and G. Sardanashvily, Phys. Rep. **94**, 1 (1983).

[13] G. Sardanashvily and O. Zakharov, Gauge Gravitation Theory, World Scientific (1992)

[14] F.W. Hehl, P. Von Der Heyde and G.D. Kerlick, Rev. Modern Phys. **48** 393 (1976).

[15] J. Nitsch and F.W. Hehl, Phys. Lett. **90B**, 98 (1980).

[16] F.W. Hehl, Y.Ne'eman, J. Nitsch and P. Von Der Heyde, Phys. Lett. **78B** 103 (1978)

[17] K. Hayashi and T. Shirafuji, Phys. Rev. **D19**, 3524 (1979).

[18] F.G. Basombrio, Gen. Relat. Grav. **12**, 109 (1980).

[19] V.C. de Andrade and J.G.Pereira, Phys. Rev. **D56**, 4689 (1997).

[20] V.C. de Andrade and J.G.Pereira, Gen. Relat. Grav. **30**, 263 (1998).

[21] A.G. Agnese and P. Calvini, Phys. Rev. **D12**, 3800 (1975).

[22] A.G. Agnese and P. Calvini, Phys. Rev. **D12**, 3804 (1975).

[23] T. Dereli and R.W. Tucker, JHEP 0203, 41 (2002).

[24] R.T. Hammond, Rep. Progr. Phys. **65**, 599 (2002).

[25] F. W. Hehl, J. D. Mc Crea, E. W. Mielke and Y. Ne'eman, Phys. Rep. **258**, 1-171 (1995).

[26] R. Tresguerres and E. W. Mielke, Phys. Rev. **D62**, 44004 (2000).

[27] I. Kirsch, Phys. Rev. **D72**, 24001 (2005).

[28] E. W. Mielke, Ann. Phys. (N. Y.) **219**, 78-108 (1992).

[29] E. W. Mielke, Phys. Rev. **D69**, 128501 (2004).

[30] F. W. Hehl and Y. N. Obukhov, arXiv:gr-qc/0508024.

[31] D. Bashkirov, G. Giachetta, L. Mangiarotti and G. Sardanashvily, J. Phys. **A38**, 5329 (2005).

[32] L. O'Raifeartaigh, Phys. Rev. **B139**, 1052 (1965).

[33] W.D. McGlinn, Phys. Rev. Lett. **12**, 467 (1964).

[34] S. Coleman and J. Mandula, Phys. Rev. **159**, 1251 (1967).

[35] L. Michel, *"Invariance in Quantum Mechanics and group extension", in Group theoretical concepts and methods in elementary particle physics*, F. Gürsey ed., Gordon & Breach, pp. 135-200 (1964).

[36] F.J. Dyson, *Symmetry Groups in Nuclear and Particle Physics*, Benjamin N.Y. (1966).

[37] O. Pelc and L. P. Horwitz, J. Math. Phys. **38**, 139 (1997).

[38] V. Aldaya, J.L. Jaramillo and J. Guerrero, J. Math. Phys. **44**, 5166 (2003).

[39] V. Aldaya, J. Navarro-Salas and A. Ramírez, Commun. Math. Phys. **121**, 541 (1989).

[40] V. Aldaya, M. Calixto, F. F. López-Ruiz and E. Sánchez-Sastre, (in preparation).

[41] V. Bargmann, Ann. Math. **59**, 1 (1954).

[42] E.J. Saletan, J. Math. Phys. **2**, 1 (1961).

[43] V. Aldaya and J. A. de Azcárraga, Int. J. Theor. Phys. **24**, 141 (1985).

[44] V. Aldaya and J. Navarro-Salas, Commun. Math. Phys. **113**, 375 (1987).

[45] J. Guerrero, J.L. Jaramillo and V. Aldaya, *Group-cohomology refinement to classify G-symplectic manifolds*, J. Math. Phys. **45** , 2051 (2004).

[46] J. Mickelsson, Phys. Rev. Lett. **55**, 2099 (1985).

[47] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Vol.2, Pergamon Press (1975).

[48] J. D. Jackson, *Classical Electrodynamics*, J. Wiley (1975).

[49] V. Aldaya, J.L. Jaramillo and J. Guerrero, J. Phys. **A35** , L1 (2002).